MATH 430, SPRING 2022
HOMEWORK , DUE FRIDAY APRIL 15

Below $\mathfrak{A} = (\mathbb{N}, <, S, +, \cdot, 0, 1)$ is the standard model of PA.

Problem 1. *Show that there is there is a nonstandard model of PA, i.e. a model $\mathcal{B} \models PA$, such that $\mathcal{B} \not\cong \mathfrak{A}$.*

Hint: Let $\mathcal{L} = \mathcal{L}_{PA} \cup \{c\}$, where c is a new constant. Define a set of sentences Γ saying that c cannot be any finite successor of 0. Show that Γ is consistent with PA.

Recall that in class we proved:

Theorem 1. *If $\mathcal{B} \models PA$, then $\mathcal{B}$ is an end extension of $\mathfrak{A}$ i.e. there is a one-to-one homomorphism $f : \mathbb{N} \to |\mathcal{B}|$ such that for all $a \in \operatorname{ran}(f)$ and $b \in |\mathcal{B}| \setminus \operatorname{ran}(f)$, $a <^{\mathcal{B}} b$.*

The homomorphism above is given by $f(n) = (S^{\mathcal{B}})^n(0^{\mathcal{B}})$, and it follows that $\mathcal{B}$ contains an isomorphic copy of the natural numbers as an initial segment. For simplicity of notation, write n to denote $(S^{\mathcal{B}})^n(0^{\mathcal{B}})$. For example, we write 0 for $0^{\mathcal{B}}$, 1 for $S^{\mathcal{B}}(0) = 1^{\mathcal{B}}$, 2 for $S^{\mathcal{B}}(S^{\mathcal{B}}(0))$ and so on.

Similarly, for a formula $\phi(x_1, ..., x_n)$ and $a_1, ..., a_n$ in $\mathbb{N}$, we say that $PA \models \phi[a_1, ..., a_n]$, if for any model $\mathcal{B} \models PA$, $\mathcal{B} \models \phi[a_1, ..., a_n]$.

With this notation, we also proved:

Theorem 2. *If $\mathcal{B} \models PA$, and $\phi(x_1, ..., x_n)$ is a Δ_0-formula, then for any $a_1, ..., a_n$ in $\mathbb{N}$, $\mathfrak{A} \models \phi[a_1, ...a_n]$ iff $\mathcal{B} \models \phi[a_1, ...a_n]$.*

You can use the above theorems for the next problems.

Problem 2. *Suppose $\mathcal{B} \models PA$.*

1. *If $\phi(x_1, ..., x_n)$ is a Σ_1-formula, show that for any $a_1, ..., a_n$ in $\mathbb{N}$, if $\mathfrak{A} \models \phi[a_1, ...a_n]$, then $\mathcal{B} \models \phi[a_1, ...a_n]$.*
2. *If $\phi(x_1, ..., x_n)$ is a Π_1-formula, show that for any $a_1, ..., a_n$ in $\mathbb{N}$, if $\mathcal{B} \models \phi[a_1, ...a_n]$, then $\mathfrak{A} \models \phi[a_1, ...a_n]$.*
3. *If $\phi(x_1, ..., x_n)$ is a Δ_1-formula, show that for any $a_1, ..., a_n$ in $\mathbb{N}$, $\mathfrak{A} \models \phi[a_1, ...a_n]$ iff $\mathcal{B} \models \phi[a_1, ...a_n]$.*

Problem 3. *Suppose $\mathcal{B} \models PA$ is a nonstandard model. Show that there is $b \in |\mathcal{B}|$ with infinitely many predecessors. Conclude that $(\mathcal{B}, <)$ is not a well order.*

Recall that the **primitive recursive functions** are (total) functions $f : \mathbb{N}^k \to \mathbb{N}$, that are built up from the constant function $f(x) = 0$, projections, and the successor function S, by applying composition and the primitive recursion operation:

- $f(0, a_1, ..., a_n) = g(a_1, ..., a_n)$;
- $f(n+1, a_1, ..., a_n) = h(n, f(n, a_1, ..., a_n), a_1, ..., a_n)$;

where g, h are primitive recursive.

Problem 4. *Prove that $f(a, b) = a + b$ is primitive recursive.*

Problem 5. *Prove that $f(a, b) = a \cdot b$ is primitive recursive.*

Problem 6. *Prove that $f(a) = 2^a$ is primitive recursive.*